Hedging volumetric risks using put options in commodity markets

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Outline

- Definitions and motivation
- Utility maximization solution
- Hedging using options
  NA approach
  V@R and Tail V@R approaches
  Case of constant volume and solution of different optimization problems
  Example for random volume

Suppose that our income is VS,

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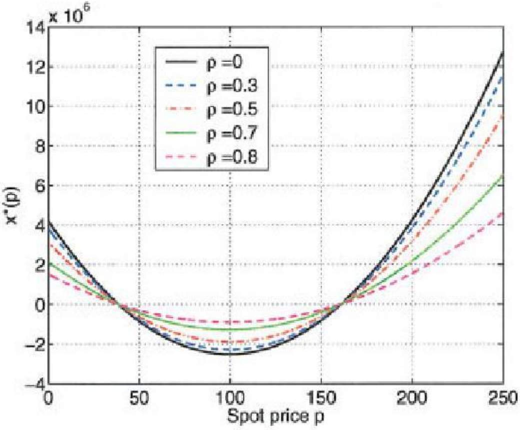
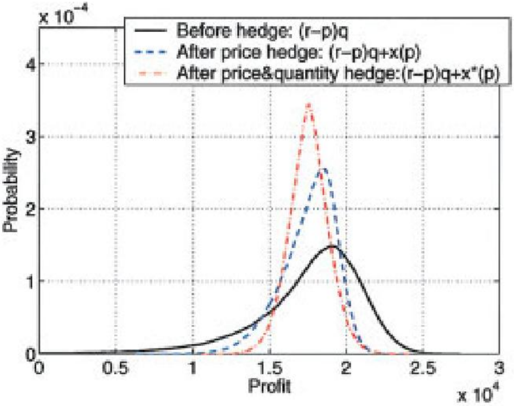
  is optimal, i.e.
- $U(X) \longrightarrow \max$, U — utility function;
- $\mathsf{E}X - \alpha\sigma(X) \longrightarrow \max$;
- $\mathrm{VaR}(X) \longrightarrow \min$ ($\operatorname{ess\,inf} X \longrightarrow \max$).

Utility maximization solution

Value function of optimal hedge $g(p)$ in the case of utility maximization ([OD06]) using CARA and joint lognormal distribution:



Distribution of income X in presence of hedge ([OD06]):



Hedging using put options, V@R and Tail V@R

Let us hedge using buying a number of put options. So the income of our portfolio will be as follows:

$$X = VS + h((K - S)^{+} - P(K)), h \geq 0.$$

Questions:

- how to maximize $V@R(X)$, $TailV@R(X)$?
- optimal h?
- optimal K?
- optimal pair (h, K)?

Definition 2.1. ([?]) Suppose $\lambda \in (0, 1]$. Consider the set

$$\mathcal{D}_\lambda = \left\{ \mathsf{Q} : \frac{d\mathsf{Q}}{d\mathsf{P}} \leq 1/\lambda \right\}.$$

Let us construct the function

$$u_\lambda(X) = \inf_{\mathsf{Q} \in \mathcal{D}_\lambda} \mathsf{E}_\mathsf{Q} X, \quad X \in L^0.$$

This is a coherent utility function. The corresponding coherent risk measure is called *Tail V@R of level* λ.

Suppose $V \equiv 1$ and S has lognormal distribution. Then due to [V11] we have that **the optimal h^* for the fixed K is as follows**:

$$h^*(K) = -\infty, \lambda > \Phi(\ln(K - P(K))),$$
$$h^*(K) = 1, \lambda \leq \min(\Phi(\ln(K - P(K))), \Phi(\ln K) - \Phi(\ln(K - P(K)))).$$

Theorem 2.4. Suppose $h \geq 0$. Also suppose that here and further NA condition is fulfilled for the model with options. Then there exists $\lambda_0 > 0$ such that for $\lambda < \lambda_0$ the optimal hedge is as follows:

$$(h^*, K^*) = (1, \infty).$$

So in this case it is optimal to sell the forward on basic asset.

Suppose $h \geq 0$. Suppose $V \equiv 1$ and $S \sim U[0,1]$. **Solution for the fixed K is as follows:**

q^*	h^*	Condition
$K - P(K)$	1	$\lambda \leq \min(K\text{-}P(K), P(K))$
$\lambda + K - 2\sqrt{\lambda P(K)}$	$\sqrt{\lambda / P(K)}$	$P(K) \leq \lambda \leq \min(K^2 / P(K), 1 - (K - P(K)))$
λ	0	$\max(K^2 / P(K), K - P(K)) \leq \lambda \leq 1 - (K - P(K))$
∞	∞	$\lambda > 1 - (K - P(K))$

As a result we have that for
$\lambda \leq \min(1 - K_0 + 2 * P(K_0), 1 - P(1), P(1))$, where

$$K_0 : P'(K_0) = 1/2,$$

the optimal hedge is as follows:

$$(h^*, K^*) = (1, 1), q^*_\lambda = 1 - P(1).$$

So in this case it is optimal to sell the forward on basic asset.

Theorems and conclusions

Theorem 3.1. Suppose V is a nonnegative random variable and S is a random variable such that for all $x \in \operatorname{supp} S$ we have that

$$(x, \operatorname{ess\,inf} V) \in \operatorname{supp}(S, V).$$

Then for all K and for measure of risk essential infinum we have the optimal hedge as follows:

$$h^*(K) = \operatorname{ess\,inf} V, \ \operatorname{ess\,inf} X^*(K) = K - P(K).$$

And also we have that the optimal $K^* = \operatorname{ess\,sup} S$.

So in this case it is optimal to sell the infinum number of forwards on basic asset.

Theorem 3.2. Suppose V is a nonnegative random variable such that

$$\mathsf{P}(V = \operatorname{ess\,inf} V) > 0$$

and S is a random variable such that

$$\operatorname{supp} S = \operatorname{supp}(S | V = \operatorname{ess\,inf} V)$$

Then for all K there exists λ_0 such that for $\lambda < \lambda_0$ the optimal hedge is as follows:

$$h^* = \operatorname{ess\,inf} V, \ \operatorname{ess\,inf} X^*(K) = K - P(K).$$

And also we have λ_0 such that for $\lambda < \lambda_0$ the optimal $K^* = \operatorname{ess\,sup} S$.

So in this case it is optimal to sell the infinum number of forwards on basic asset.

Example with random volume

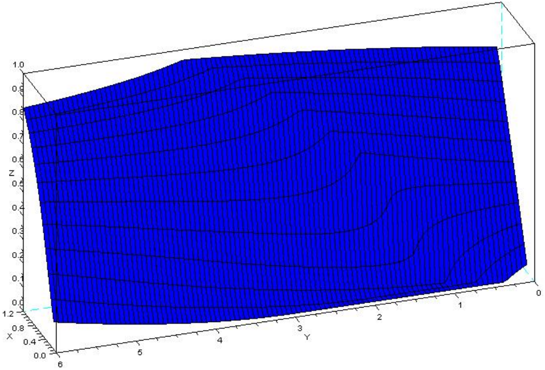
Consider the following example:
Let $V \sim U[1, b], S \sim U[0, 1]$ and they are independent.
Then due to the Theorem 3.1. we have that for all K and for measure of risk essential infinum we have the optimal hedge as follows:

$$h^*(K) = \operatorname{ess\,inf} V, \; \operatorname{ess\,inf} X^*(K) = K - P(K).$$

And also we have that the optimal $K^* = 1$.
So in this case it is optimal to sell the infinum number of forwards on basic asset.
Let us for fixed K define

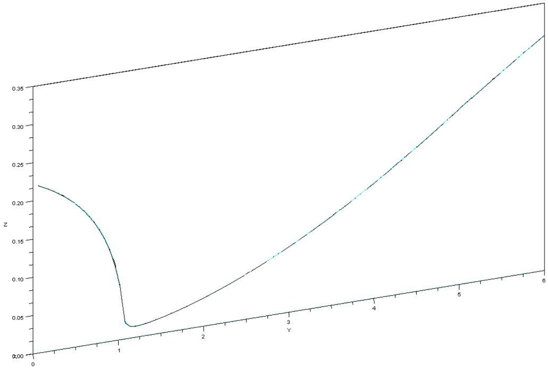
$$\lambda(K)(a, h) = \mathrm{P}(VS + h((K - S)^+ - P(K)) \leq a).$$

The picture for $\lambda(K)(a, h)$



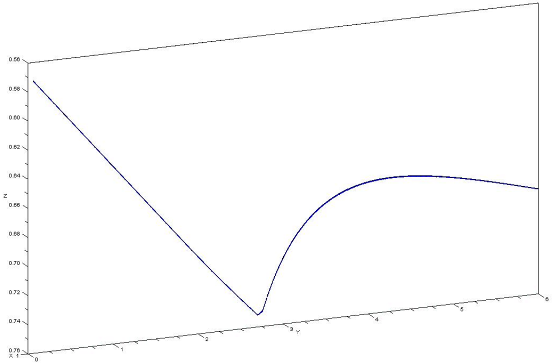
The picture for $\lambda(K)(a_0, h)$ for small $a_0 \sim K - P(K)$ is as follows:
So for

$$\lambda = \lambda_0, q^*_\lambda(h) = a_0, h^* = h_0 \sim 1.$$



The picture for $\lambda(K)(a_1, h)$ for large a is as follows:
So for

$$\lambda = \lambda_1, q^*_\lambda(h) = a_1, h^* = \infty.$$



Hedging using put options: under investigation.

- options with different K's:

$$X = VS + \sum_i h_i((K_i - S)^+ - C(K_i))$$

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- position with some assets:

$$X = V(S_1 + S_2) + \sum_i h_i((K_i - S_i)^+ - C_i(K_i))$$

  - importance of the choice of K_i:

$$\mathrm{P}(S_i < q_i) = 1\% \;\Rightarrow\; P(S_1 < q_1,\; S_2 < q_2) \sim 0.1\%$$

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  - options on sum $S_1 + S_2$
- hedging using expected shortfall.

- Various optimization problems for hedging quantity risk.
- Hedging using V@R and put options in the case of non random volume.
- Some results and theorems for non random case.
- Example of random volume and numeric and graphical solutions for hedging quantity risk in this case (there are nontrivial ones).

Thank you for your attention

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